Performance Evaluation and Networks

Discrete time
Markov Chains (MC)

Discrete time Markov Chain (MC): definition

Xn = the state at time n

Process $(X_n)_{n\in\mathbb{N}}$ where X_n r.v. over $(\Omega, \mathscr{F}, \mathbb{P})$, with values in E.

Definition (Markov Chain : MC)

(X_n) *markovian* *if* $\forall n \in \mathbb{N}$, $\forall x_0, \ldots, x_n, x_{n+1} \in E$ (*space of states*),

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n)$$

subject to $\mathbb{P}(X_n = x_n, \ldots, X_0 = x_0) \neq 0$.

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⚠ Stochastic process

All r.v. X_n are defined over the same probabilistic space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in the same space E → each realization $\omega \in \Omega$ yields a trajectory $X_0(\omega), X_1(\omega), X_2(\omega), \ldots$ within E.

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⚠ Convention for conditional probas

All formulas from the course with conditional probas are valid only if well defined: $\mathbb{P}(A|B)$ well defined if $\mathbb{P}(B) \neq 0$.
With this convention, the note “subject to ...” will be omitted for now, but stay alert in practice.

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Intuition: "future only depends on present", "memoryless", ...

homogeneous iff the transition. probabilities are independent of the time t

Definition (Time Homogeneous MC : HMC)

(X_n) *homogeneous* *if* $\forall n \in \mathbb{N}, \forall i, j \in E, \mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i)$

directed edge from i to j

Definition (Transition matrix & graph of Homogeneous MC)

- *Transition matrix*: $P = (p_{ij})_{i,j \in E}$ *with* $p_{ij} = \mathbb{P}(X_1 = j | X_0 = i)$
- *Transition graph*: *vertices* $= E$, *edge* $i\ j$ *if* $p_{ij} > 0$ (*weight* p_{ij})

Discrete Time Markov Chain (MC): examples ?

Discrete Time Markov Chain (MC): examples

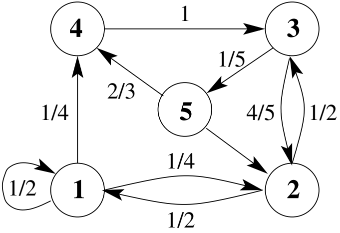
- Jeu de l’oie / Snakes and ladders
- Sequence of i.i.d. r.v. for any law over E.
- Uniform random walk over $\mathbb{N}^d$ or $\mathbb{Z}^d$.
- Some randomized algorithms, e.g. in system/network protocols.

Proposition (Characteristic example)

Let $(U_n)_{n \in \mathbb{N}^*}$ i.i.d. sequence of r.v. with values in F, E finite or countable space, f map $E \times F \to E$, X_0 r.v. with values in E and independent of the sequence (U_n), then the recurrence equation $X_{n+1} = f(X_n, U_{n+1})$ define an homogeneous MC with values in E.

Transition matrix & graph

	1	2	3	4	5
1	1/2	1/4	0	1/4	0
2	1/2	0	1/2	0	0
3	0	4/5	0	0	1/5
4	0	0	1	0	0
5	0	1/3	0	2/3	0



P = stochastic matrix:

- positive coeff: $\forall i, j,\ p_{ij} \geq 0$
- $\sum$ over line = 1 : $\forall i, \sum_j p_{ij} = 1$

MC = "random walk" :

realization $X_0(\omega), X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega), X_5(\omega), \ldots$:

walk in the transition graph

Important Notation: let $i \in E$, $\mathbb{P}_i(A) \stackrel{\text{def}}{=} \mathbb{P}(A|X_0 = i)$ for event A

$\mathbb{E}_i(Z) \stackrel{\text{def}}{=} \mathbb{E}(Z|X_0 = i) = \sum_z z \cdot \mathbb{P}(Z = z|X_0 = i)$ for real r.v. Z ($\sum$ou$\int$)

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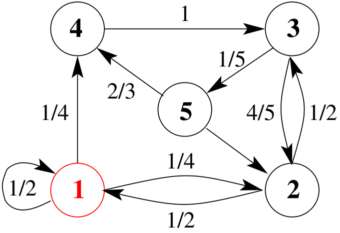

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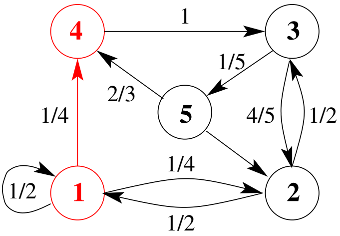

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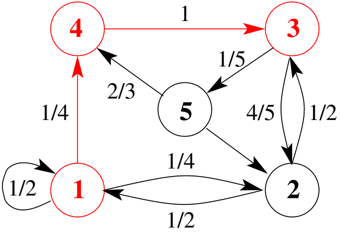

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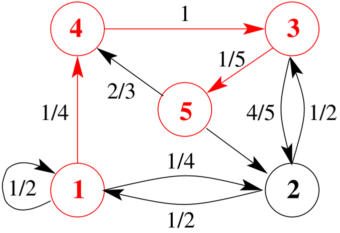

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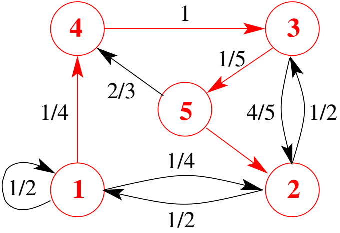

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Markov property in practice

Theorem ("General" Markov property)

Let (X_n) *MC with values in* E*, at time* $n \in \mathbb{N}$ *in state* $i \in E$*,*
let $I^+ \in \mathscr{P}(E)^{\otimes\mathbb{N}}$ *a set of trajectories in the future,*
let $I^- \in \mathscr{P}(E^n)$ *a set of trajectories in the past,*
$\mathbb{P}((X_{n+1}, X_{n+2}, \dots) \in I^+ | (X_0, \dots, X_{n-1}) \in I^-, X_n = i) = \mathbb{P}((X_{n+1}, X_{n+2}, \dots) \in I^+ | X_n = i)$
And if homogeneous MC, this term is: $= \mathbb{P}((X_1, X_2, \dots) \in I^+ | X_0 = i)$

English formulation: $\forall i \in E$, $\forall n \in \mathbb{N}$, the future at time n and the past at time n are conditionally independent given the present state $X_n = i$.

Examples of use:

$\mathbb{P}(X_{10} = a, X_7 = b | X_5 = c, X_3 = d, X_2 = e) = \mathbb{P}(X_{10} = a, X_7 = b | X_5 = c)$

$\mathbb{P}(\forall n \geq 11, X_n \notin \{a, b\} | X_{10} = c, \forall n \leq 9, X_n \in \{d, e\}) = \mathbb{P}(\forall n \geq 11, X_n \notin \{a, b\} | X_{10} = c)$

Chapman-Kolmogorov Equations (I)

Notation: $p_{ij}(r, r+s) \stackrel{\text{def}}{=} \mathbb{P}(X_{r+s} = j | X_r = i)$ for $i, j \in E$, $r, s \in \mathbb{N}$.

Theorem (Chapman-Kolmogorov)

Any MC $(X_n)_{n \in \mathbb{N}}$ satisfies the equations: $\forall i, j, k \in E$, $\forall r, s, t \in \mathbb{N}$,

$$p_{ij}(r, r+s+t) = \sum_k p_{ik}(r, r+s) p_{kj}(r+s, r+s+t)$$

Corollary (Matrix version)

matrix multiplication

Given matrices $P(r, r+s) \stackrel{def}{=} \left(p_{ij}(r, r+s)\right)_{i,j \in E}$, *then* $\forall r, s, t \in \mathbb{N}$,

$$P(r, r+s+t) = P(r, r+s) P(r+s, r+s+t)$$

Corollary (Homogeneous case)

If HMC, proba to jump from i to j in n steps = coeff i,j of P^n denoted $p_{ij}(n)$.

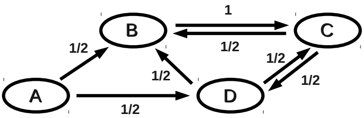
Chapman-Kolmogorov Equations (II)

Vector notation of the law ν of r.v. X with values in E:

$$\nu = (\nu_i)_{i \in E} \text{ line vector with } \nu_i \stackrel{\text{def}}{=} \mathbb{P}(X = i)$$

Corollary (Homogeneous case)

If HMC, the law $\pi^{(n)}$ of X_n is fully set by the matrix P and the law $\pi^{(0)}$ of X_0: $\pi^{(n)} = \pi^{(0)} P^n$.



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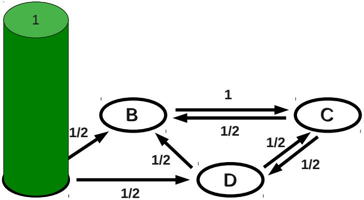
\pi^(n) is the evolution of the probabilities at every state at time n



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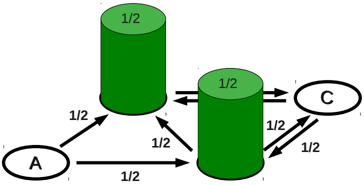

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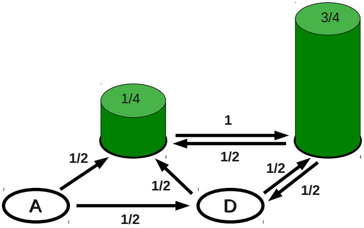

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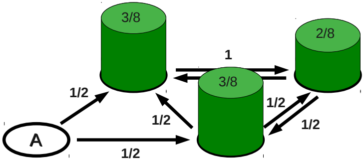

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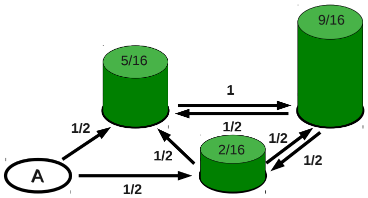

Stopping time : definition & examples

Definition (Stopping time of a stochastic process)

Stopping time T of stoch proc $(X_n)_{n\in\mathbb{N}}$: r.v. with values in $\mathbb{N}\cup\{+\infty\}$ s.t. $\forall n\in\mathbb{N}$, event $\{T=n\}$ can be described using $X_0,\ldots,X_n$: $\{T=n\}=\{(X_0,\ldots,X_n)\in I\}$ for a set of trajectories $I\subseteq E^{n+1}$.

Intuition: time event which can be expressed with no reference to the future.

Examples: let (X_n) MC with values in E and $F\subseteq E$,

- Time to reach F : $\tau_F=\inf\{n\geq 0|X_n\in F\}$?
- Time to come back to F : $T_F=\inf\{n\geq 1|X_n\in F\}$?
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the third item is not a stopping time, because it depends on the future

small difference between the

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Special notation: for $i\in E$, $T_i\overset{\text{def}}{=}T_{\{i\}}$ and $\tau_i\overset{\text{def}}{=}\tau_{\{i\}}$.

Stopping time: quick exercise

Exercise: let T, T_1, T_2 stopping times for (X_n), tell whether the next r.v. are also stopping times for (X_n) ?

1. a constant r.v. c
2. $T + c$ where $c \in \mathbb{N}^*$ fixed
3. $T - c$ where $c \in \mathbb{N}^*$ fixed
4. $\min(T_1, T_2)$
5. $\max(T_1, T_2)$
6. $N(t) = \max\{n \in \mathbb{N} | X_0 + X_1 + \ldots + X_n \leq t\}$ (X_n positive r.v.)
7. $N(t) + 1$

Strong Markov property: regeneration

Homogenity is important in MC

In homogeneous MC, what's important is that what happens now does not deppe

Theorem (Strong Markov property)

- *Let T stopping time for HMC (X_n), then subject to $T < +\infty$ and $X_T = i$, $(X_{T+n})_{n\geq 0}$ is markovian and independent of $X_0, \dots, X_T$ (also denoted $(X_{T\wedge n})_{n\geq 0}$ où $\wedge = \min$).*
- *Moreover, for any event A described with $X_0, \dots, X_T$ and $I^+ \in \mathscr{P}(E)^{\otimes \mathbb{N}}$*

$$\mathbb{P}((X_{T+1}, X_{T+2}, \dots) \in I^+ | X_T = i, T < +\infty, A) = \mathbb{P}((X_1, X_2, \dots) \in I^+ | X_0 = i)$$

I+ is the sequence of trajectories in the future

Intuition: starting to look at some HMC from a stopping time = reset counters to zero

⚠ if T not a stopping time, risk to lose this property (cf TD).

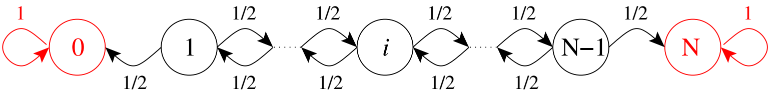
⚠ if MC not homogeneous, risk to lose this property (even if T stopping time).

"One step forward": small step without strong Markov (I)

\tau_F is time to reach F

Example: probability $\mathbb{P}_i(\tau_F < +\infty)$ to reach a set F of states starting from state i

Application: non biased walk over $\{0, \ldots, N\}$ where $0, N$ absorbing



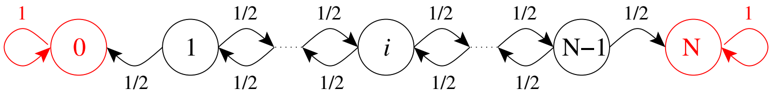
"One step forward": small step without strong Markov (I)

Example: probability $\mathbb{P}_i(\tau_F < +\infty)$ to reach a set F of states starting from state i

Proposition

The values $h_i = \mathbb{P}_i(\tau_F < +\infty)$ form the minimum positive solution in $\mathbb{R}$ of the linear system:
$$\begin{cases} h_i = 1 & \text{for all } i \in F \\ h_i = \sum_{j\in E} p_{ij} h_j & \text{for all } i \notin F \end{cases}$$

Application: non biased walk over $\{0,\ldots,N\}$ where $0,N$ absorbing

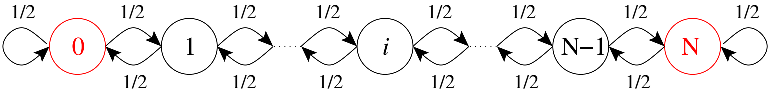


"One stepforward method" :

"One step forward": small step with strong Markov (II)

Example: Mean time $\mathbb{E}_i(\tau_F)$ to reach a set F of states starting from state i

Application: 1D non biased walk over $\{0, \ldots, N\}$ with $F = \{0, N\}$.



“One step forward”: small step with strong Markov (II)

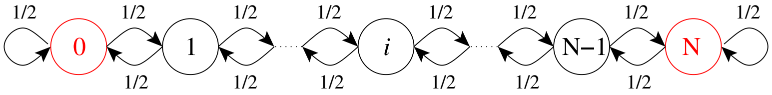
Example: Mean time $\mathbb{E}_i(\tau_F)$ to reach a set F of states starting from state i

Proposition

The values $t_i = \mathbb{E}_i(\tau_F)$ form the min positive solution in $\mathbb{R} \cup \{\infty\}$ of the linear system:

$$\begin{cases} t_i = 0 & \textit{pour tout } i \in F \\ t_i = 1 + \sum_{j \notin F} p_{ij} t_j & \textit{pour tout } i \notin F \end{cases}$$

Application: 1D non biased walk over $\{0, \ldots, N\}$ with $F = \{0, N\}$.



“One step forward”: big step with strong Markov

Example: law of nb of visits to state i given reaching probabilities, for HMC (X_n).

Lemma (nb of visits to a state & probas of access between states)

Let $N_i \overset{def}{=} \sum_{n=1}^{+\infty} \mathbb{1}_{X_n=i}$ *nb of visits to i from time 1,*
Let $f_{ij} \overset{def}{=} \mathbb{P}_i(T_j < \infty)$ *proba de reach j after leaving i,*
Then :

$$\mathbb{P}_j(N_i = n) = \begin{cases} f_{ji} f_{ii}^{n-1}(1-f_{ii}) & \text{if } n \geq 1 \\ 1 - f_{ji} & \text{if } n = 0 \end{cases}$$

Corollary (returns to the same state)

If $f_{ii} = 1$, *then* $\mathbb{P}_i(N_i = \infty) = 1$ *et* $\mathbb{E}_i(N_i) = +\infty$.
If $f_{ii} < 1$, *then* $\mathbb{P}_i(N_i = \infty) = 0$ *et* $\mathbb{E}_i(N_i) = f_{ii}/(1-f_{ii}) < +\infty$.

binary probability (cannot have a probability with value in (0,1))

Irreducibility: definitions

Definition (Communication in HMC)

Two states i et j communicate if there exist a path from i to j and a path from j to i in the transition graph. (directed paths)

Proposition (Classes of communication)

Communication = equivalence relation partionning states into equiv classes, called classes of communication (= strongly connected components of the transition graph).

Definition (Irreducible HMC)

HMC is irreducible if it has only one class of communication (i.e. strongly connected transition graph).

Irreducibility: structure

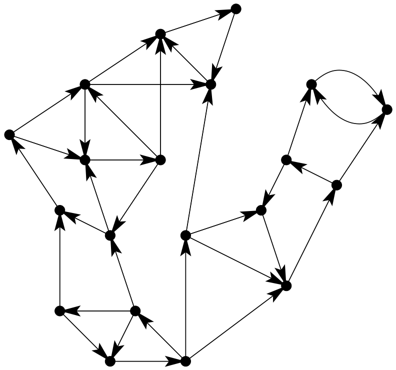
Proposition (Bags with no cycle)

Let G directed graph, with strongly connected components $C_1, \ldots, C_p$*, then its* quotient graph *(for strong connection relation) defined by* $\langle G \rangle = G/C_1/\ldots/C_p$ *(contraction of each component into one vertex) is acyclic.*

Definition (Closed/final/absorbing class)

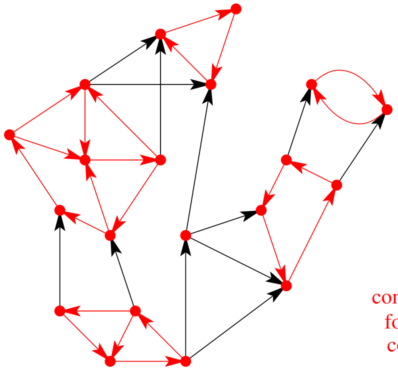
Class of communication is ***closed/final/absorbing*** *if all states reachable from this class remain in this class ("maximal" strongly connected comp. in the quotient graph).*

Irreducibility: example



⚠ if nb ∞ of states, one may see ∞ classes or classes ∞.

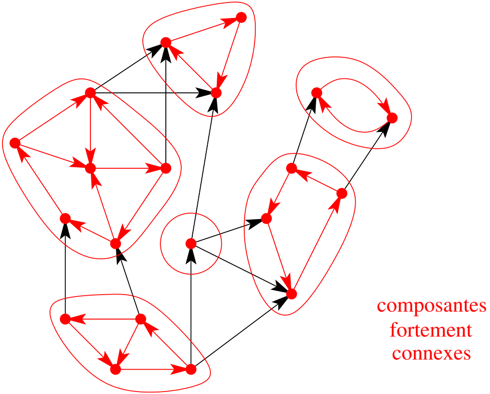
Irreducibility: example



composantes fortement connexes

⚠ if nb ∞ of states, one may see ∞ classes or classes ∞.

Irreducibility: example



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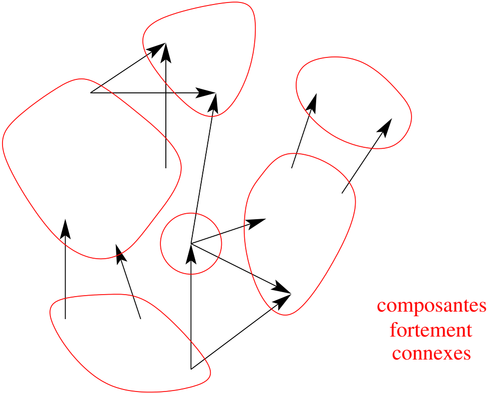
Irreducibility: example



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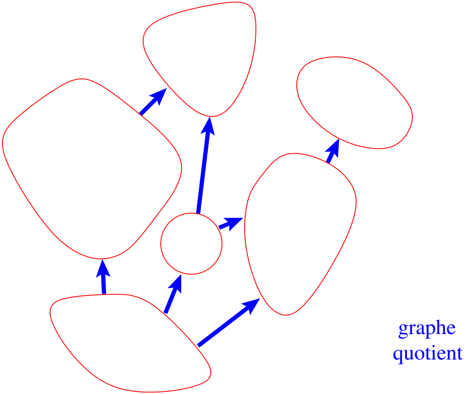
Irreducibility: example



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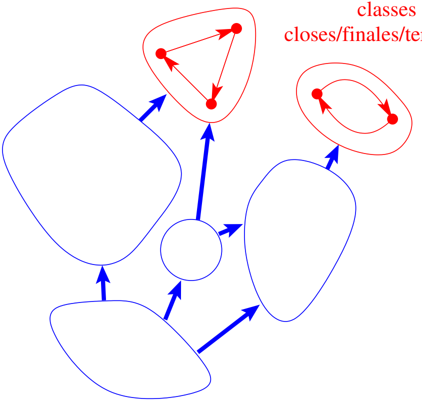
Irreducibility: example



⚠ if nb ∞ of states, one may see ∞ classes or classes ∞.

Periodicity: definitions

Definition (Period of a state in HMC)

State i has period $d_i \overset{def}{=} \mathrm{GCD}\{n \geq 1 | p_{ii}(n) > 0\}$ *(i.e.* GCD *lengths of cycles traversing i in the transition graph).*

Proposition (Irreducibility & periodicity)

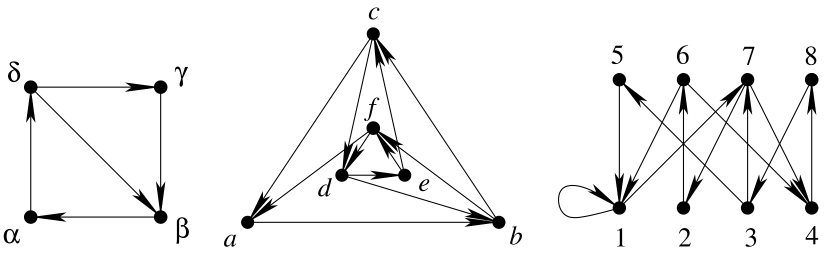
In a class of communication (strong. conn. comp.), all states have the same period.

Definition (Period of an irreducible HMC)

▷ *Period of irred HMC: period common to all its states*
(= PGCD *lengths of all cycles in transition graph.*
▷ *Aperiodic irred HMC: if period = 1.*

Periodicity: examples

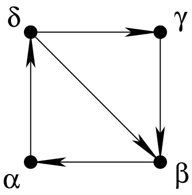
Exercise: find the period of those graphs.



A **B** **C**

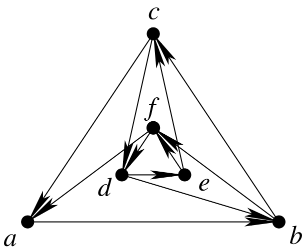
Periodicity: examples

Exercise: find the period of those graphs.

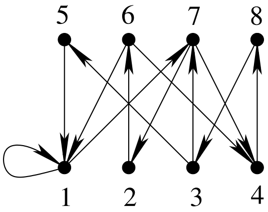


A

period = 1



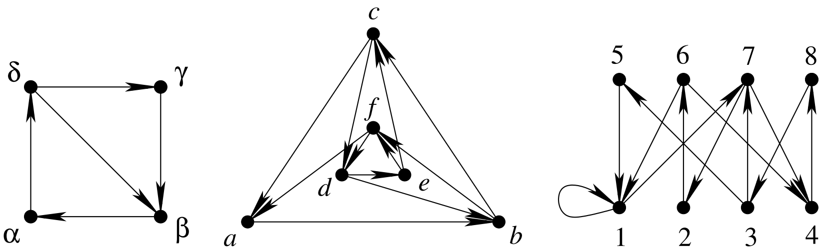
B



C

Periodicity: examples

Exercise: find the period of those graphs.



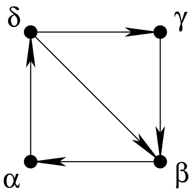
A period = 1

B period = 3

C

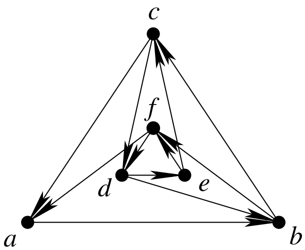
Periodicity: examples

Exercise: find the period of those graphs.



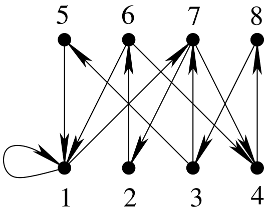
A

period = 1



B

period = 3



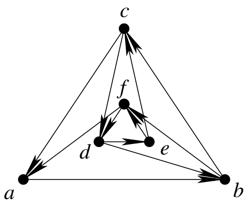
C

period = 1

Periodicity: structure

Theorem (cycle of bags)

Let G strongly connected directed graph of period d, then there exists a partition $V_0,\dots,V_{d-1}$ of vertices such that any edge leaving V_p reaches V_{p+1} (with the convention $V_{d+1} = V_0$).



Periodicity: structure

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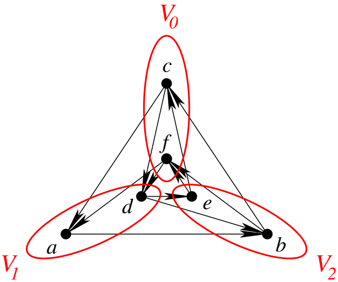


Invariance: definitions

Framework: (X_n) HMC with transition matrix P.

Definition (Invariant/stationnary measure)

Invariant/stationnary measure for P: $\mu = (\mu_i)_{i \in E} \in \mathbb{R}^E$ such that $\mu \geq 0$, $\mu \neq 0$ and $\mu P = \mu$, i.e. $\forall i\ \mu_i \geq 0$, $\exists i\ \mu_i \neq 0$ and $\sum_j \mu_j p_{ji} = \mu_i$.

Definition (Invariant/stationnary probability distribution)

Inv./stat. distribution for P : invariant measure μ with $\sum_{i \in E} \mu_i < +\infty$. In this case, renormalized $\pi = (\pi_i)_{i \in E}$ with $\pi_i = \mu_i / \sum_{j \in E} \mu_j$ is called invariant/stationnary probability distribution ($\sum_{i \in E} \pi_i = 1$).

Terminology: if law of X_n = invariant proba distrib, the process is said to be “in stationnary regime”, “at equilibrium” ...

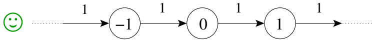
Invariance: structure

Exercise: how many invariant proba distrib for an HMC ?

0	?
1	?
nb fini ≥ 2	?
∞	?

Invariance: structure

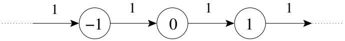
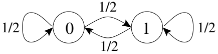
Exercise: how many invariant proba distrib for an HMC ?

- 0 

- 1 ?
- nb fini ≥ 2 ?
- ∞ ?

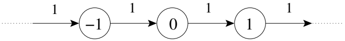
Invariance: structure

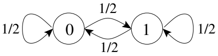
Exercise: how many invariant proba distrib for an HMC ?

0	☺ 
1	☺ 
nb fini ≥ 2	?
∞	?

Invariance: structure

Exercise: how many invariant proba distrib for an HMC ?

- 0 ☺ 

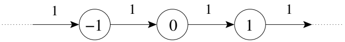
- 1 ☺ 

- nb fini ≥ 2 ☹ Impossible
- ∞ ?

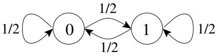
Invariance: structure

Exercise: how many invariant proba distrib for an HMC ?

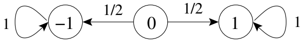
- 0 — ☺



- 1 — ☺



- nb fini ≥ 2 — ☹ Impossible
- ∞ — ☺



Theorem (structure of invariant proba distrib)

The invariant proba distrib of an HMC form a convex polyhedron in $\mathbb{R}_+^E$: it is the convex hull of the invariant proba distrib of final classes of communication.

If we have two invariants proba distr, then everyting in between is also an invariant prob

because any convex combination of two invariant measures is also invariant

Recurrence: definitions

All MC that is reccurent is positive reccurent. The case of null recurrent can only happen when the #states is infinite.

Definition (transitory/recurrent null/positive state)

Let (X_n) HMC with values in E and T_i time to return to i,

- *state i transitory if* $\mathbb{P}_i(T_i < +\infty) < 1$, there is a positive proba that some
- *state i recurrent if* $\mathbb{P}_i(T_i < +\infty) = 1$,
- *state i null recurrent if* $\mathbb{P}_i(T_i < +\infty) = 1$ *but* $\mathbb{E}_i(T_i) = +\infty$,
- *state i positive recurrent if* $\mathbb{E}_i(T_i) < +\infty$ *thus* $\mathbb{P}_i(T_i < +\infty) = 1$.

Proposition (finite return time ⇔ infinite nb of visits)

state i recurrent ⇔ $\mathbb{P}_i$*(∞ nb of visits of i)=1* ⇔ $\mathbb{E}_i$*(nb of visits of i)=+∞*
state i transitory ⇔ $\mathbb{P}_i$*(finite nb of visits of i)=1* ⇔ $\mathbb{E}_i$*(nb of visits of i)<+∞*

Corollary (potential matrix criterium)

$$i \text{ recurrent iff} \sum_{n=0}^{+\infty} p_{ii}(n) = +\infty$$

Irreducibility & Recurrence

Proposition

In a class of communication (strong. conn. comp.) of an HMC, the states are either all recurrent, or all transitory.
If the are recurrent, the class is closed and $\forall j, \mathbb{P}(T_j < +\infty) = 1$.

Corollary

An irreducible chain is either recurrent *(all states are recurrent), or* transitory *(all states are transitory).*

Question: HMC irreducible ⇒ HMC recurrent ?

Irreducibility & Recurrence

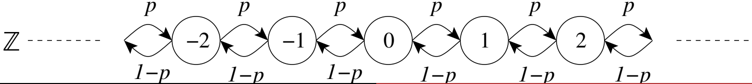
Proposition

In a class of communication (strong. conn. comp.) of an HMC, the states are either all recurrent, or all transitory.
If the are recurrent, the class is closed and $\forall j, \mathbb{P}(T_j < +\infty) = 1$.

Corollary

An irreducible chain is either recurrent *(all states are recurrent), or* transitory *(all states are transitory).*

Question: HMC irreducible ⇒ HMC recurrent ? NO !
Contrex : 1D walk space homogeneous, recurrent iff $p = 1/2$
(compute $p_{00}(n)$ explicitely then estimate $\sum_{n=0}^{+\infty} p_{ii}(n)$ with Stirling)



Invariance & Recurrence

Theorem (if irreducible, recurrence ⇒ invariant measure)

Let (X_n) HMC irred and recurrent, of transition matrice P,
Let state 0 fixed arbitrarily and T_0 time to return to 0,
Let $V_i \stackrel{def}{=} \sum_{n=1}^{T_0} \mathbb{1}_{X_n=i}$ nb of visits of i between time 0 (excluded) and return time T_0 (included), define $x_i \stackrel{def}{=} \mathbb{E}_0[V_i]$ average nb of visits of i between two visits of 0. Then:

1. $0 < x_i < \infty$ *for all* $i \in E$
2. $(x_i)_{i\in E}$ *invariant measure of* P (canonical inv measure for 0)
3. P *admits an unique invariant measure up to a constant factor*

⚠ HMC irreducible, with invariant measure ⇒ HMC recurrent ?

Invariance & Recurrence

Theorem (if irreducible, recurrence ⇒ invariant measure)

Let (X_n) HMC irred and recurrent, of transition matrice P, Let state 0 fixed arbitrarily and T_0 time to return to 0, Let $V_i \stackrel{def}{=} \sum_{n=1}^{T_0} \mathbb{1}_{X_n=i}$ nb of visits of i between time 0 (excluded) and return time T_0 (included), define $x_i \stackrel{def}{=} \mathbb{E}_0[V_i]$ average nb of visits of i between two visits of 0. Then:

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2. $(x_i)_{i\in E}$ *invariant measure of* P (canonical inv measure for 0)
3. *P admits an unique invariant measure up to a constant factor*

⚠ HMC irreducible, with invariant measure ⇒ HMC recurrent ?
NO ! look again 1D space homogeneous random walk, $p \neq 1/2$, they admit $\mathbf{1} = (\ldots,1,1,1,\ldots)$ as invariant measure Recall that the 1D homogeneous random walk i

Invariance & Positive recurrence

Theorem (if irreducible, positive recurrence ⇔ inv proba distrib)

Let (X_n) HMC irred, of transition matrix P, we have the equivalence:

1. (X_n) *admits a positive recurrent state,* being "only recurrent" is not enough
2. (X_n) *has all its states positive recurrent,*
3. (X_n) *admits an invariant proba distribution.*

In this case, the invariant proba distrib $\pi = (\pi_i)$ *is unique and satisfies* $\pi_i = 1/\mathbb{E}_i(T_i) > 0$ *where* T_i *time to return to* i*. The chain is called* positive recurrent.

Ex of HMC irred recurrent but not positive recurrent ?

\pi_i is the time spent in state_i.If Ei(Ti) is large, i.e. the average time need

Invariance & Positive recurrence

Theorem (if irreducible, positive recurrence ⇔ inv proba distrib)

Let (X_n) HMC irred, of transition matrix P, we have the equivalence:

1. *(X_n) admits a positive recurrent state,*
2. *(X_n) has all its states positive recurrent,*
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In this case, the invariant proba distrib $\pi = (\pi_i)$ is unique and satisfies $\pi_i = 1/\mathbb{E}_i(T_i) > 0$ where T_i time to return to i. The chain is called positive recurrent.

Ex of HMC irred recurrent but not positive recurrent ? YES, e.g. symmetric random walk over $\mathbb{Z}$!

Special case: HMC with finite nb of states

Proposition

any finite state irreducible HMC is positive recurrent.

Theorem (Perron 1907 - Frobenius 1912)

Let P transition matrix of irred HMC, with N states, with period d, with sorted complex eigenvalues $|\lambda_1| \geq \ldots \geq |\lambda_N|$ then

1. $\lambda_1 = 1$ *eigenvalue of* P,
2. *complex unit roots* $\lambda_1 = \omega^0, \lambda_2 = \omega^1, \ldots, \lambda_d = \omega^{d-1}$ *où* $\omega = e^{2\pi i/d}$, *are eigenvalues of* P,
3. *other eigenvalues* $\lambda_{d+1}, \ldots, \lambda_N$ *satisfy* $|\lambda_j| < 1$.

\pi is the invariant distribution

\lambda_2 is the second largest eigen value

Corollary (irred and aperiodic HMC)

$P^n = \mathbf{1}^T \pi + O(n^{m_2 - 1} |\lambda_2|^n)$ *where* m_2 *multiplicity of* λ_2 $(|\lambda_2| < 1)$

Asymptotic convergence:

This theorem gives information about the asymptotic behavior of the Markov Chain

Theorem (Convergence in law for HMC)

Let (X_n) *HMC* irreducible, positive recurrent, aperiodic, *of transition matrix* P *and stationary distribution* π. *Then for any initial distribution* ν, *for any state* i,

stationary = invariant

$$\lim_{n \to +\infty} \mathbb{P}(X_n = i) = \pi_i$$

More precisely, $\lim_{n \to +\infty} ||\nu P^n - \pi||_\infty = 0.$

this is stronger than the one above

A classical proof: by coupling Markov chains

⚠ Essential hypothesis: period = 1.

Ergodic theorem for HMC

Theorem (Ergodicity for HMC)

Let (X_n) HMC with values in E, irred, positive recurrent of invariant distrib π, and let $f : E \to \mathbb{R}$ such that $\sum_{i \in E} |f(i)|\pi_i < \infty$, then for any initial law ν, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \sum_{i\in E} f(i)\pi_i$$

"intuitively", on the left hand side, we kind of make a statistic

Periodic irreducible case

Question: dealing with irred HMC of period $d \geq 2$?

Periodic irreducible case

Question: dealing with irred HMC of period $d \geq 2$?

Reductions: return to aperiodic case with $\frac{I+P+\cdots+P^{d-1}}{d}$ or P^d

Theorem (Convergence - periodic case)

Let (X_n) *HMC* irreducible, positive recurrent, of period d, *with transition matrix* P, *let* $V_0, \ldots, V_{d-1}$ *the bag cycle partition. Then for any initial distribution* ν, *for all* $0 \leq r \leq d-1$, *for any state* $i \in V_r$,

$$\lim_{n \to +\infty} \mathbb{P}(X_{nd+r} = i) = d / \mathbb{E}_i(T_i)$$

More precisely, $\lim_{n \to +\infty} ||\nu P^{nd+r} - d/\mathbb{E}_i(T_i)||_\infty = 0$

Periodic irreducible case

Question: dealing with irred HMC of period $d \geq 2$?

Reductions: return to aperiodic case with $\frac{I+P+\cdots+P^{d-1}}{d}$ or P^d ⚠ loosing irred

Theorem (Convergence - periodic case)

Let (X_n) *HMC* irreducible, positive recurrent, of period d, *with transition matrix* P, *let* $V_0, \ldots, V_{d-1}$ *the bag cycle partition. Then for any initial distribution* ν, *for all* $0 \leq r \leq d-1$, *for any state* $i \in V_r$,

$$\lim_{n\to+\infty} \mathbb{P}(X_{nd+r} = i) = d/\mathbb{E}_i(T_i)$$

More precisely, $\lim_{n\to+\infty} ||\nu P^{nd+r} - d/\mathbb{E}_i(T_i)||_\infty = 0$

Non irreducible case

Asymptotic study of the general case

- Study the transition graph structure and identify final classes
- Study the absorption probabilities of each final class
- Study the asymptotic behaviour in each final class (period, recurrence, invariant distribution ...)